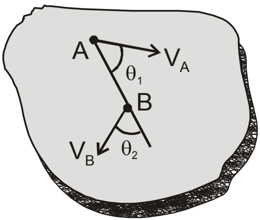
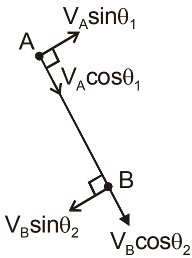
RIGID BODY DYNAMICS

1. RIGID BODY :

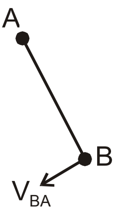


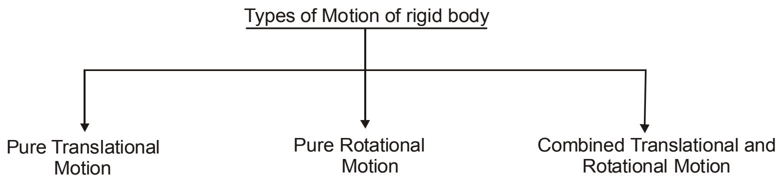


If the above body is rigid

$$V_A \cos\theta_1 = V_B \cos\theta_2$$

V_{BA} = relative velocity of point B with respect to point A.





2. MOMENT OF INERTIA (I) :

Definition : Moment of Inertia is defined as the capability of system to oppose the change produced in the rotational motion of a body.

Moment of Inertia is a scalar positive quantity.

$$I = m r_1^2 + m_2 r_2^2 + \dots\dots\dots\dots\dots\dots\dots\dots$$
$$= I_1 + I_2 + I_3 + \dots\dots\dots\dots\dots\dots\dots\dots$$

SI units of Moment of Inertia is Kgm^2.

Moment of Inertia of :

2.1 A single particle : $I = mr^2$

where m = mass of the particle

r = perpendicular distance of the particle from the axis about which moment of Inertia is to be calculated

2.2 For many particles (system of particles) :

$$I = \sum_{i=1}^{n} m_i r_i^2$$

2.3 For a continuous object :

$$I = \int dmr^2$$

where dm = mass of a small element

r = perpendicular distance of the particle from the axis

2.4 For a larger object :

$$I = \int dI_{element}$$

where dI = moment of inertia of a small element

3. TWO IMPORTANT THEOREMS ON MOMENT OF INERTIA :

3.1 Perpendicular Axis Theorem

[Only applicable to plane lamina (that means for 2-D objects only)].

$I_z = I_x + I_y$ (when object is in x-y plane).

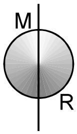
3.2 Parallel Axis Theorem

(Applicable to any type of object):

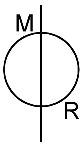
$I_{AB} = I_{cm} + Md^2$

List of some useful formula :

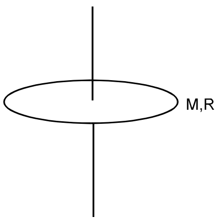
Object	Moment of Inertia



Solid Sphere — $\frac{2}{5}MR^2$ (Uniform)

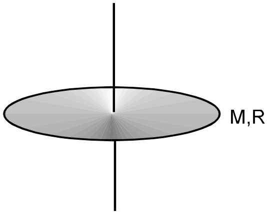


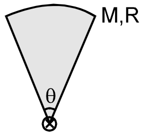
Hollow Sphere — $\frac{2}{3}MR^2$ (Uniform)



MR^2 (Uniform or Non Uniform)

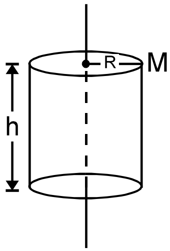
Ring.

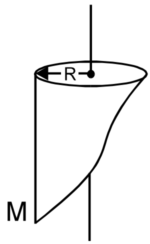




$\frac{MR^2}{2}$ (Uniform)

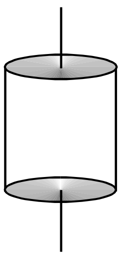
Disc





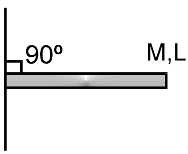
MR^2 (Uniform or Non Uniform)

Hollow cylinder

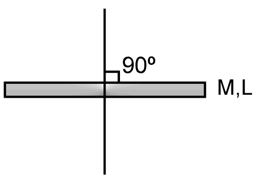


$\frac{MR^2}{2}$ (Uniform)

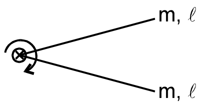
Solid cylinder



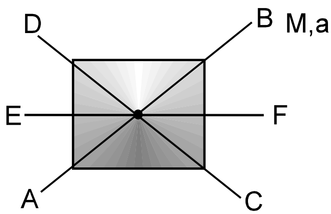
$\frac{ML^2}{3}$ (Uniform)



$\frac{ML^2}{12}$ (Uniform)

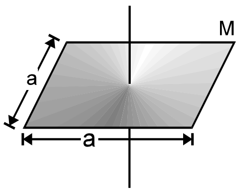


$$\frac{2m\ell^2}{3} \text{ (Uniform)}$$



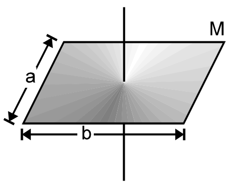
Square Plate

$$I_{AB} = I_{CD} = I_{EF} = \frac{Ma^2}{12} \text{ (Uniform)}$$



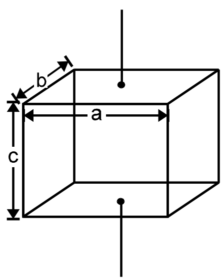
$$\frac{Ma^2}{6} \text{ (Uniform)}$$

Square Plate



$$I = \frac{M(a^2 + b^2)}{12} \text{ (Uniform)}$$

Rectangular Plate



$$\frac{M(a^2 + b^2)}{12} \text{ (Uniform)}$$

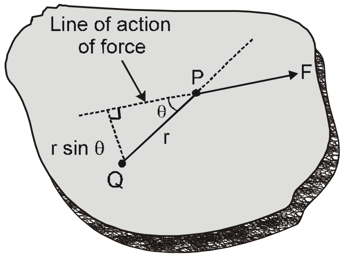
Cuboid

4. RADIUS OF GYRATION :

$$I = MK^2$$

5. TORQUE :

$$\vec{\tau} = \vec{r} \times \vec{F}$$

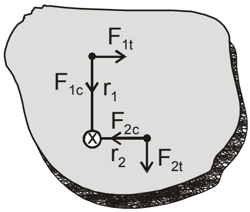


5.5 Relation between 'τ' & 'α' (for hinged object or pure rotation)

$$\vec{\tau}_{ext}\Big)_{Hinge} = I_{Hinge}\ \vec{\alpha}$$

Where $\vec{\tau}_{ext}\Big)_{Hinge}$ = net external torque acting on the body about Hinge point

I_{Hinge} = moment of Inertia of body about Hinge point



$$F_{1t} = M_1a_{1t} = M_1r_1\alpha$$
$$F_{2t} = M_2a_{2t} = M_2r_2\alpha$$
$$\tau_{resultant} = F_{1t}\,r_1 + F_{2t}\,r_2 + \ldots\ldots\ldots$$
$$= M_1\,\alpha\,r_1^{\,2} + M_2\,\alpha\,r_2^{\,2} + \ldots\ldots\ldots\ldots$$
$$\tau_{resultant}\Big)_{external} = I\,\alpha$$

Rotational Kinetic Energy = $\frac{1}{2}.I.\omega^2$

$$\vec{P} = M\vec{v}_{CM} \quad \Rightarrow \quad \vec{F}_{external} = M\vec{a}_{CM}$$

Net external force acting on the body has two parts tangential and centripetal.

$$\Rightarrow \quad F_C = ma_C = m\frac{v^2}{r_{CM}} = m\omega^2 r_{CM} \quad \Rightarrow \quad F_t = ma_t = m\alpha\, r_{CM}$$

6. **ROTATIONAL EQUILIBRIUM :**

For translational equilibrium.

$$\Sigma F_x = 0 \quad \text{.............} \quad (i)$$

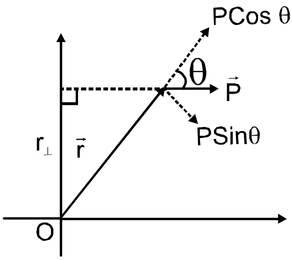
and $$\Sigma F_y = 0 \quad \text{.............} \quad (ii)$$

The condition of rotational equilibrium is

$$\Sigma \Gamma_z = 0$$

7. **ANGULAR MOMENTUM ($\vec{L}$)**

7.1 Angular momentum of a particle about a point.



$\vec{L} = \vec{r} \times \vec{P}$ $\Rightarrow$ L = rpsinθ

$$|\vec{L}| = r_\perp \times P$$

$$|\vec{L}| = P_\perp \times r$$

7.3 Angular momentum of a rigid body rotating about fixed axis :

$$\vec{L}_H = I_H \vec{\omega}$$

L_H = angular momentum of object about axis H.
I_H = Moment of Inertia of rigid object about axis H.
ω = angular velocity of the object.

7.4 Conservation of Angular Momentum

Angular momentum of a particle or a system remains constant if $\tau_{ext} = 0$ about that point or axis of rotation.

7.5 Relation between Torque and Angular Momentum

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

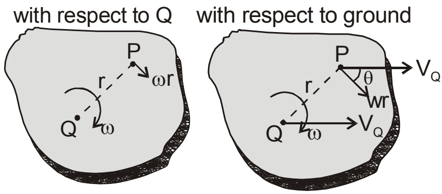
Torque is change in angular momentum

7.6 Impulse of Torque :

$$\int \tau dt = \Delta J \qquad \Delta J \rightarrow \text{Change in angular momentum.}$$

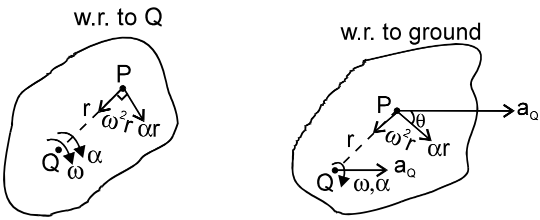
For a rigid body, the distance between the particles remain unchanged during its motion i.e. $r_{P/Q}$ = constant

For velocities



$$V_P = \sqrt{V_Q^{\,2} + (\omega r)^2 + 2\,V_Q\,\omega r \cos\theta}$$

For acceleration :



θ, ω, α are same about every point of the body (or any other point outside which is rigidly attached to the body).

Dynamics :

$$\vec{\tau}_{cm} = I_{cm}\,\vec{\alpha}\ ,\ \vec{F}_{ext} = M\vec{a}_{cm}$$

$$\vec{P}_{system} = M\vec{v}_{cm}\ ,$$

Total K.E. $= \dfrac{1}{2}Mv_{cm}^2 + \dfrac{1}{2}I_{cm}\,\omega^2$

Angular momentum axis AB = $\vec{L}$ about C.M. + $\vec{L}$ of C.M. about AB

$$\vec{L}_{AB} = I_{cm}\,\vec{\omega} + \vec{r}_{cm} \times M\vec{v}_{cm}$$